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Reconstructing ternary Dowling geometries

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Abstract

We prove that the only geometry of rank $n > 4$ all of whose proper contractions are ternary Dowling geometries is the ternary Dowling geometry. We use this to prove a stronger version of a conjecture of Joseph Kung and James Oxley.

Keywords: Matroid; Dowling geometry; Signed graph

1. Introduction

Dowling geometries [2–4] play an important role in a number of areas of matroid theory [6,9,10,12]. In this paper we focus on ternary Dowling geometries, proving a result stronger than a conjecture of Kung and Oxley [8] that grew out of some of their work in extremal matroid theory.

Before sketching the elements in the work of Kung and Oxley that also play a role in this paper, we briefly define some matroids to be considered. Fix a basis $p_1, p_2, \ldots, p_n$ of the rank-n ternary projective geometry $PG(n-1, 3)$. The rank-n ternary Dowling geometry

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Q_n is, up to isomorphism, the restriction of $PG(n-1,3)$ to the set of points on *coordinate lines*:

$$E(Q_n) = \bigcup_{1 \leqslant i < j \leqslant n} \mathrm{cl}\big(\{p_i, p_j\}\big).$$

The elements $p_1, p_2, \ldots, p_n$ are the *joints* of Q_n, while the other points of Q_n are its *internal points*. Note that Q_n has n^2 points. The ternary Reid geometry R_9 is the restriction of $PG(2,3)$ to the union of three concurrent lines, two having four points and the other having three. Much of the interest in R_9 lies in the fact that it is representable over a field F if and only if F has characteristic 3.

Kung [7] proved that if a rank-n geometry (simple matroid) G is representable over $GF(3)$ and $GF(q)$, where q is a prime power not divisible by 3, then G contains at most n^2 points. Its proof rests on another theorem involving the Reid geometry R_9:

Theorem 1.1. *If a ternary geometry G contains a point x for which $|G| - |G/x| \geqslant 2r(G)$, then G contains the Reid geometry R_9 as a minor.*

In other words, if G is a ternary geometry that does not contain R_9 as a minor, then $|G| - |G/x| \leqslant 2r(G) - 1$ for all points x of G. Hence a rank-n ternary geometry with no R_9-minor has at most n^2 points. In the sequel to [7], Kung and Oxley [8] investigated rank-n geometries representable over both $GF(3)$ and $GF(q)$ with exactly n^2 points. They proved the following theorem.

Theorem 1.2. *Let G be a ternary geometry of rank $n \geqslant 3$. If G does not have R_9 as a minor and contains n^2 points, then G is isomorphic to the ternary Dowling geometry Q_n when $n > 3$, and either G is isomorphic to Q_3 or the ternary affine plane $AG(2,3)$ when $n = 3$.*

Since the Reid geometry R_9 is representable over a field F if and only if F has characteristic 3, Theorem 1.2 implies that a rank-n ($n \geqslant 3$) geometry G representable over $GF(3)$ and $GF(q)$ with n^2 points is isomorphic to Q_n if $n > 3$ and to Q_3 or $AG(2,3)$ if $n = 3$. The main step in their proof uses upper homogeneity, that is, if G is a rank-n geometry containing n^2 points and all proper upper intervals of G are ternary Dowling geometries, then G must also be a ternary Dowling geometry. They noted that this result is reminiscent of Kantor's embedding theorem [5] for projective spaces, and it led them to the following conjecture:

Conjecture 1.3. *There exists an integer k such that for any $n > k$ and any ternary geometry G of rank n for which every upper interval of rank k is isomorphic to the ternary Dowling geometry Q_k, we must have G embeddable in Q_n.*

The class of subgeometries of ternary Dowling geometries is closed under taking minors; it is the same class as that of bias matroids on signed graphs [6,13]. A natural problem is to determine the excluded minors for the class [13]. Kung and Oxley [8] observed that, by using the Scum Theorem, Conjecture 1.3 is a weakened version of the problem of finding excluded minors for the class of bias matroids on signed graphs.

In this paper, we will prove the following theorem, which is a strengthening of Conjecture 1.3.

Theorem 1.4. *Let G be geometry of rank $n > 4$ such that every upper interval of G of rank 4 is isomorphic to the rank-4 ternary Dowling geometry Q_4. Then G is isomorphic to the ternary Dowling geometry Q_n.*

Let M be a geometry of rank $n \geqslant 3$ such that every upper interval of rank 3 is isomorphic to $M(K_4)$. Aigner proved that, with three exceptions, M is isomorphic to $M(K_{n+1})$ [1]. Since the cycle matroids of complete graphs are Dowling geometries over the trivial group and the ternary Dowling geometries are Dowling geometries over the group with two elements, Theorem 1.4 can be viewed as an extension of Aigner's result.

Instead of proving Theorem 1.4 directly, we shall prove the following theorem.

Theorem 1.5. *Let G be a geometry of rank $n > 4$ such that the contraction G/x is isomorphic to the ternary Dowling geometry Q_{n-1} for all points x of G. Then G is isomorphic to Q_n.*

The excluded minors for the class of ternary matroids are $U_{2,5}$, $U_{3,5}$, F_7, and F_7^*. So if a geometry G is not ternary, then G contains a minor N isomorphic to one of these four. Then by the Scum Theorem, G must have an upper interval with a minor isomorphic to N. But if a geometry G satisfies the conditions in Theorems 1.4 and 1.5, then it does not contain any of $U_{2,5}$, $U_{3,5}$, F_7, or F_7^* as a minor, and so must be representable over $GF(3)$. Thus, in the proofs of Theorems 1.4 and 1.5, we are free to use the fact that G is ternary.

Theorem 1.5 does not hold when $n = 4$: there are two ternary geometries M, with 13 and 14 points respectively, such that the contraction M/x is isomorphic to Q_3 for all x of M. The proof involves tedious case analysis and is omitted.

Without confusion, we can assume a contraction of a geometry is the simplification of the contraction, which is again a geometry. Points, lines, and planes of a geometry are its flats of rank 1, 2, and 3, respectively. For a geometry G, $|G|$ will denote the number of points in G. We shall say that k points of G *vanish* by the contraction of a point x if $|G| - |G/x| = k$.

2. Preparation

We first establish that Theorem 1.4 follows from Theorem 1.5. For now we assume that Theorem 1.5 holds and argue by induction on n. Let G be a rank-n ($n > 4$) geometry satisfying the conditions in Theorem 1.4. Then $G \cong Q_5$ follows trivially when $n = 5$, so suppose $n > 5$ and the theorem holds for smaller values. Let x be any point of G and let $G' = G/x$. If N is an upper interval of G' of rank 4, then there exists points $x_1, x_2, \ldots, x_{n-5}$ of G' such that $N = G'/\{x_1, x_2, \ldots, x_{n-5}\}$. But then $N = G/\{x, x_1, x_2, \ldots, x_{n-5}\}$ is an upper interval of G of rank 4. That is, every upper interval of G' of rank 4 is isomorphic to Q_4. By induction, $G' \cong Q_{n-1}$. That is, for every point x of G, the contraction G/x is isomorphic to Q_{n-1}. Now G satisfies the condition of Theorem 1.5, which then implies Theorem 1.4.

Before we proceed to the proof of Theorem 1.5, we need some preliminaries. If a rank-n geometry G satisfies the condition of Theorem 1.5, then by the Scum Theorem, every rank-k flat of G is isomorphic to a subgeometry of Q_k. In particular, every plane of G is isomorphic to a subgeometry of Q_3.

There are three non-isomorphic rank-3 ternary geometries with nine points [7], namely the affine plane $AG(2,3)$, the Reid geometry R_9, and the Dowling geometry Q_3. This becomes clear if we consider their complements in the projective plane $PG(2,3)$: a subgeometry of $PG(2,3)$ with four points is isomorphic to a 4-point line, or the union of a 3-point line and a point not on the line, or a circuit of size 4, respectively.

Let r be the common point of the two 4-point lines in R_9. Then $R_9 \setminus r$ is isomorphic to $AG(2,3) \setminus y$ for any point y of $AG(2,3)$. Further $R_9 \setminus r$ is not isomorphic to a subgeometry of Q_3 since it has no 4-point line and any subgeometry of Q_3 with eight points contains at least one 4-point line. The deletion from R_9 of any other point is isomorphic to a subgeometry of Q_3. If G satisfies the condition of Theorem 1.5, then G has no minor isomorphic to $R_9 \setminus r$. Now we only need to consider the subgeometries of Q_3.

It is easy to see the only possible non-isomorphic subgeometries with eight points are the deletion of a joint and the deletion of an internal point from Q_3. We will denote by P_1 the deletion of a joint and by P_2 the deletion of an internal point of Q_3. Since P_1 contains one and P_2 contains two 4-point lines, P_1 and P_2 are not isomorphic.

To catalog the non-isomorphic subgeometries of Q_3 with seven points, we consider the two points x and y that are deleted from Q_3. If two joints are deleted from Q_3, then the geometry is isomorphic to the non-Fano plane F_7^-; if x is a joint and y an internal point such that x and y are not contained in the same coordinate line, then the geometry is isomorphic to P_7; if x and y are both internal points that are contained in a coordinate line, then the geometry is isomorphic to the parallel connection of two 4-point lines that we will denote by N_7. Denote by O_7 the geometry obtained by deleting a joint x and an internal point y that are contained in the same coordinate line. If x and y are internal points not contained in the same coordinate line, then the geometry is isomorphic to O_7. These are the only non-isomorphic subgeometries of Q_3 with seven points.

Any subgeometry with six points is isomorphic to either the cycle matroid $M(K_4)$, the rank-3 whirl $\mathcal{W}^3$, the 2-sum R_6 of two 4-point lines, or the parallel connection of a 4-point line and a 3-point line. See [11] for geometric representations of some of the geometries we have described.

Now we give a brief description of the flats of Dowling geometries (see [3] for details). A rank-k flat of Q_n is isomorphic to the direct sum $Q_i \oplus M(K_{i_1+1}) \oplus M(K_{i_2+1}) \oplus \cdots \oplus M(K_{i_t+1})$, where $i, i_1, i_2, \ldots, i_t$ are non-negative integers with sum k. In particular, the lines have two, three, or four points and a plane is isomorphic to one of the following: a set of three independent points $M(K_2)$, the direct sum of a 3-point line $M(K_3)$ and a point $M(K_2)$, the direct sum of a 4-point line Q_2 and a point $M(K_2)$, $M(K_4)$, or Q_3. So if a plane of Q_n with at least six points is not isomorphic to $M(K_4)$, then it is isomorphic to Q_3. In general, if a rank-k ($k > 3$) flat of Q_n contains more than $(k-1)^2 + 1$ points, then it is isomorphic to Q_k. For easy reference, we list these facts as a lemma.

Lemma 2.1. *Let F be a flat of Q_n. Then*

(i) *If F is a plane with at least six points that is not isomorphic to $M(K_4)$, then F is isomorphic to Q_3.*
(ii) *If F has rank k ($k > 3$) and has more than $(k-1)^2 + 1$ points, then F is isomorphic to Q_k.*

3. The proof of Theorem 1.5

Suppose G is a rank-n ($n > 4$) ternary geometry such that $G/x \cong Q_{n-1}$ for all points x of G. To prove Theorem 1.5, we first prove through a sequence of lemmas that G has a flat isomorphic to Q_k for each rank k, $4 \leqslant k \leqslant n-1$.

Lemma 3.1. *For every $k \geqslant 3$, any flat of G of rank k is isomorphic to a subgeometry of Q_k. Furthermore, if a flat is isomorphic to Q_k for some k, then F is a modular flat of G.*

Proof. Let F be a flat of G of rank $k < n$. Then for any point x not in F, the restriction of G to F is contained as a subset of a rank-k flat in $G/x \cong Q_{n-1}$. Therefore F, as a geometry, is contained in Q_k. By the hypothesis of Theorem 1.5, if G has a flat isomorphic to a ternary Dowling geometry of rank at least 3, then it is a modular flat of G. $\square$

Lemma 3.2. *The geometry G is 3-connected.*

Proof. It is obvious that G is connected. If G is connected but not 3-connected, then G is a 2-sum or a parallel connection of two proper minors of G, say N_1 and N_2. At least one of N_1 and N_2, say N_1, has rank at least three since the sum of the ranks of N_1 and N_2 is $n + 1 > 5$. Then there exists a point x of N_1 such that G/x is a 2-sum of N_1/x and N_2, which is not 3-connected, contradicting the fact that Q_{n-1} is 3-connected. $\square$

Lemma 3.3. *Let M be a rank-k ($k \geqslant 4$) subgeometry of Q_k. If $M/x \cong Q_{k-1}$ for all points x of M, then $M \cong Q_k$.*

Proof. By Lemma 3.2 M is 3-connected. First we show that all joints $p_1, p_2, \ldots, p_k$ are contained in M. Assume not. Without loss of generality, say $p_1 \notin M$. Then for some distinct pair i and j, $2 \leqslant i, j \leqslant k$, since M is 3-connected, there is a point x of M that is an internal point on the line spanned by p_i and p_j. But then M/x will be a proper subgeometry of Q_{k-1}, contradicting $M/x \cong Q_{k-1}$. Now assume M contains all the joints. If an internal point is not contained in M, say an internal point on the line spanned by p_1 and p_2 is missing, then for $p_3 \in M$, M/p_3 is not isomorphic to Q_{k-1}. $\square$

Lemma 3.4. *There is a plane P of G that contains at least six points and is not isomorphic to $M(K_4)$. Furthermore, if G has a 4-point line, then G has a plane with at least seven points.*

Proof. We first prove that G has a line that contains at least three points. This certainly holds if G has a 4-point line. So assume that G has no 4-point lines. Take any point x of G. Since $G/x \cong Q_{n-1}$, there exists a 4-point line in G/x. Hence G has a plane containing x with at least five points. If this plane has no 3-point line, then it contains a subset isomorphic to the uniform matroid $U_{3,5}$. This contradicts the fact that G is ternary. So G has a line l with at least three points. Suppose y is a point contained in the line l. In the contraction G/y, the line l becomes a point and it is contained in a 4-point line of G/y. That is, G has a plane P with at least six points that contains l; furthermore, when contracting y, the plane P becomes a 4-point line. Since every contraction of $M(K_4)$ is a 3-point line, the plane P is not isomorphic to $M(K_4)$. □

Denote by m for the maximum number of points in a plane of G. Then $6 \leqslant m \leqslant 9$. Suppose that P is a plane of G with m points and that P is not isomorphic to $M(K_4)$. By Lemma 3.4, if G has a 4-point line, then P at least seven points.

Lemma 3.5. *There is a rank-4 flat F of G containing P such that $F \cong Q_4$.*

Proof. From the description of flats of Dowling geometries in Section 2, all planes of a ternary Dowling geometry that are not isomorphic to $M(K_4)$ and contain at least six points are isomorphic to Q_3. Since P has at least six points and is not isomorphic to $M(K_4)$, P is contained as a subset in a coordinate plane in G/x for every point x not in P. If P has nine points, then certainly $P \cong Q_3$ by Lemma 3.1. Suppose P has eight points. Then P is isomorphic to P_1 or P_2. Since $|P_i| - |P_i/x| = 5$ for some point x in each case for $i = 1, 2$, there is a point y of P such that $|P| - |P/y| = 5$. That is, five points of P vanish when one contracts to P/y, which is a line contained in a coordinate plane of $G/y \cong Q_{n-1}$. Therefore, G has a flat F of rank four that contains P with $|F| \geqslant 9 + 5 = 14$ points. By Lemma 3.1, the flat F is isomorphic to a subgeometry of Q_4 and does not contain R_9 as a minor. By Theorem 1.1, at most $2r(F) - 1 = 7$ points vanish in the contraction of F by any point. Thus the contraction of F by any point contains at least $14 - 7 = 7$ points. By Lemma 2.1, it is isomorphic to Q_3, because it is a plane of Q_{n-1} with at least seven points. Now Lemma 3.3 applies and we have $F \cong Q_4$. Consequently, $P \cong Q_3$.

Now assume that P contains six or seven points. By hypothesis G has a rank-4 flat F containing P in which the contraction F/v by any point $v \in F \setminus P$ is isomorphic to Q_3. Hence F contains at least 10 points. Since $(F/u)/v = (F/v)/u \cong U_{2,4}$ for any point u of P, F/u contains a 4-point line as a minor. We prove by contradiction that F/u is not isomorphic to the direct sum of $U_{2,4}$ and a point w. Assume $F/u \cong U_{2,4} \oplus w$. First we assume that P is not isomorphic to N_7. Since P/u is a line with at least three points for any point u of P, in the contraction F/u, P becomes a subset of $U_{2,4}$. This forces all points off P to the point w in F/u, i.e., u and all points of F off P are collinear. Thus $F/v \cong P$ for any point $v \in F \setminus P$, contradicting that $F/v \cong Q_3$. Now suppose P is isomorphic to N_7 and let x be the common point of the two 4-point lines. Then for any point $u \neq x$ of P, using the argument above, F/u contains $U_{2,4}$ as a minor and F/u is not isomorphic to the direct sum of $U_{2,4}$ and a point. Recall the contraction F/x contains $U_{2,4}$ as a minor. Assume that F/x is isomorphic to the direct sum of $U_{2,4}$ and a point w. Then one of the two 4-point lines of P becomes a point in F/x. This implies that F is the parallel connection of a 4-point line and a plane, contradicting the fact that F/v is isomorphic to Q_3 by any point $v \in F \setminus P$. Hence F/x contains $U_{2,4}$ as a minor, and F/x is not isomorphic to the direct sum of $U_{2,4}$ and a point. By Lemma 2.1, F/u is isomorphic to Q_3 since it is a rank-3 flat of $G/u \cong Q_{n-1}$. Now F satisfies the condition of Lemma 3.3, so $F \cong Q_4$. $\square$

We are ready to prove Theorem 1.5 by induction on n.

First suppose $n = 5$. Then the flat $F \cong Q_4$ in Lemma 3.5 is a hyperplane of G. For any point x of F, the contraction G/x makes at least 7 points vanish, so G contains at least 7 points off F. By Lemma 3.1, the flat F is a modular hyperplane in G, so any line spanned by a pair of points in $G \setminus F$ meets F in a point. That is, a pair of points in $G \setminus F$ corresponds to a unique point of F. There are at least $\binom{7}{2} = 21$ such pairs and F contains 16 points, so at least two pairs are corresponding to the same point w for some point w in F. Hence w is either contained in a 4-point line l_1 or w is contained in two 3-point lines l_2 and l_3, where each of l_1, l_2 and l_3 is not contained in F. But then at least $7+2=9$ points vanish in the contraction G/w, and consequently G contains at least $16+9=25$ points. Since G is ternary and does not contain R_9 as a minor, G contains at most $5^2 = 25$ points by Theorem 1.1. Hence G contains exactly 25 points. By Theorem 1.2, $G \cong Q_5$.

Now assume $r(G) > 5$. Let $F \cong Q_4$ be a flat of G by Lemma 3.5, then $F/x \cong Q_3$ is contained in a rank-4 coordinate flat of $G/x \cong Q_{n-1}$ for any point x of F. Since the contraction F/x makes seven points vanish, G has a rank-5 flat F' containing F such that F' contains at least $16+7=23$ points. The flat F' is isomorphic to a subgeometry of Q_5. So by Theorem 1.1, $|F'| - |F'/x| \leqslant 2r(F') - 1$ for all points x of F', since F' does not contain R_9 as a minor. Thus $|F'/x| \geqslant |F'| - 2r(F') + 1 \geqslant 23 - 10 + 1 = 14$ for all points x of F'. By Lemma 2.1, any rank-4 flat of a Dowling geometry with more than 10 points is isomorphic to Q_4. So $F'/x \cong Q_4$ for all points x of F'. By Lemma 3.3, $F' \cong Q_5$.

Repeating this argument we can assume that G has a hyperplane H isomorphic to Q_{n-1}. For any point x of H, $2(n-2)+1 = 2n-3$ points vanish in the contraction H/x, so G contains at least $2n-3$ points outside H. Now H is a modular flat in G and every line of G meets H. For each pair of points off H, the line spanned by the pair meets H in a point. There are $\binom{2n-3}{2}$ pairs and there are $(n-1)^2$ points in H. Since $n > 5$, $\binom{2n-3}{2} > (n-1)^2$ and at least two pairs are corresponding to the same point y in H for some point y of H. Now the contraction G/y makes at least $2n-1$ points vanish. Hence G contains at least $(n-1)^2 + 2n - 1 = n^2$ points. Since G does not contain R_9 as a minor, G contains at most n^2 points by Theorem 1.1. Therefore G contains exactly n^2 points and $G \cong Q_n$ by Theorem 1.2.

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